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Mathematical Model of the Dynamics of Psychotherapy

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Abstract: *In this paper we introduce a mathematical model of psychotherapy where the proposed continuous dynamical system describes the relationship between the client and the therapist based on dyadic interactions modeling. The model also incorporates the influence representing the external environment of the client's state. It was assumed that given that influence is negative. In order to reflect possible instability of the client's state, we describe the client who can demonstrate other mental comorbidities. Additionally, this assumption depicts the fact that his/her preferable state can be not unique. We analyse basic properties of the model and use it to study various scenarios of final results of psychotherapy.*

Key Words: psychotherapy, dyadic interactions, dynamical system, local and global stability, instability

INTRODUCTION

Starting from the 1990s, according to Vallacher and Nowak (1994) dynamical systems became an increasingly common tool used to describe interactions between two (or more) people. However, we owe the use of such systems about therapy to Gottman and Murray, who pioneered the use of discrete dynamic systems in prediction of a marital breakdown, as presented in the articles by Gottman, Swanson, and Murray (1990) and Gottman, Murray, Swanson, Tyson, and Swanson (2002), respectively.

Although the literature on mathematical approaches describing the effects of psychotherapy is not fully extensive compared to conventional analytical approaches, two different ways in using dynamical systems modelling of psychotherapy can be distinguished. The former method is based on the idea of describing emotions (emotional states or valences) in the relation therapist-client what we can found in Baker, Peluso, Freund, Diaz, and Ghaness (2022), Liebovitch, Peluso, Norman, Su, and Gottman (2011), Peluso, Liebovitch, Gottman, Norman, and Su (2012) and Strawinska-Zanko, Liebovitch, and Peluso (2018). The latter one is related to the so-called common factors appearing in various psychotherapy schools (Imel & Wampold 2008). Schiepek, Aas, and Viol (2016) proposed a differential dynamic system based on modeling therapy

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components, such as motivation, success, problems, emotions and new perspectives. They emphasized the importance of the aforementioned components for possible changes and development during the therapy. By analyzing changes in the patient's and therapist's emotions, it can be better understood what kind of processes occur during the therapeutic sessions and what the effects are. Scholler, Viol, Aichhorn, Hütt, and Schiepek (2018) added variables that indicate changes in experiences, emotions and behaviors. Moreover, Schiepek et al. (2016) proposed to use mathematical and statistical methods to analyze therapeutic data, which allow for more accurate understanding of the dynamics of the therapeutic process. Their models describe psychotherapy as a process of self-organization of biopsychosocial systems. The solutions' aim is to predict and explain discontinuous changes in psychotherapy, using real-time high-frequency psychotherapy that affects integration into everyday practice (Schiepek, Tominschek & Heinzel 2014; Schiepek, Aichhorn, Gruber, Strunk, Bachler, & Aas 2016). The results confirm the possibility of monitoring the effects of regular therapy in routine psychotherapeutic conditions. Daily collection of psychological surveys allows the analysis of nonlinear features of therapeutic change processes, such as pattern transitions or critical instabilities.

Another method for modeling psychological processes is the Fokker-Planck model of psychotherapy, which can be formulated in one or two dimensions, engaging probabilistic differential equations. The book Tschacher and Haken (2019) outlines two minimal models for both dimensions. In the one-dimensional minimal model, the therapist's influence on the client's states is formalized. In this model, the therapist's personality significantly impacts the client's states. Therefore, it is preferable for the therapist to be less susceptible to random events, and the connection between the client and the therapist should be stronger than the opposite one. The two-dimensional minimal model enables modeling of the therapist-client relationship through a limit cycle or periodic attractor. The structural-mathematical modeling of the process of psychotherapy is ended by defining a two-dimensional Fokker-Planck equation.

We would like to emphasize that although stochastic models seem to be more realistic as all processes in nature contain many random elements, but at the end we are interested in various kinds of means resulting from the model, which can be also described using deterministic models. Yet another point of view can be found in Gelo and Salvatore (2016) who framed an approach to therapy that is heavily influenced by nonlinear dynamics rather than a direct implementation of differential equations to psychotherapy transcripts. Additionally, in Mellado, Guastello, Martınez, Tomicic, and Krause (2023) a symbolic dynamics algorithm was used to identify the unique content of therapist-client interactions.

Models specifically related to suicide can be found in Bryan and Rudd (2018), where the authors presented research based on suicidal thoughts among soldiers in the American army. They have applied models based on the cusp catastrophe. The results of the study indicate that the emotional instability of the client is crucial in considering the success of therapy.

In literature, modeling interpersonal relationships using a dynamical

system described by differential equations focuses mainly on presenting romantic relationships based on known love stories (Rinaldi et al., 2015). An interesting example describing relationships based on the novel "Pride and Prejudice" can be found in Rinaldi, Della Rossa and Landi (2014). In a broader aspect, the model of relationships between two people or groups of people (called actors in this context) can be found in Liebovitch et al. (2011) and became the basis for further research.

In this paper we describe the effects of psychotherapy using a continuous dynamical system reflecting interactions between a client and her/his therapist. The model considered by us is intentionally minimal, because we define the client's mood, therapist's and environmental state as one-dimensional variables, whereas it is commonly known that emotionality is higher-dimensional due to six or even more basic emotions. However, we would like to highlight the novelty of our approach. Firstly, we have added a new equation describing the influence of the external environment, which is crucial in the context of therapy effectiveness. This equation allows us to better understand how environmental factors affect the course of therapy and what its implications are. Secondly, we have introduced a new function that takes into account possible mental comorbidity leading to the emotional instability of the client. This is an important component because the emotional state of the patient can be crucial to the success of therapy. Thirdly, a new coefficient that indicates the level of therapist's experience is added. In our work, we also conducted simulations which assumed the negative impact of the external environment as the most difficult case of therapy. Our results help us to understand the challenges that may arise in therapy and how they can be effectively overcome.

MODEL

In this part, we introduce a model of the dynamics of psychotherapy, which we formulate on the basis of the model proposed by Leibovitch et al. (2011). The differences between the two models will be indicated at the end of this section.

Let $C(t)$ denote the state of a client at time t. We describe emotional state/valencies of the person who needs help. The simplest equation describing the rate of changes in the client's mood reads

$$\dot{C} = -m_C C + b_C$$

where $m_C > 0$, $b_C \in \mathscr{R}$ mean respectively the parameter of "internal" changes of the client's mood and his/her preferable emotional state in solitude (Peluso et al., 2012). This means that in solitude, the client's state will always tend to b_C / m_C after any disturbance from it. Moreover, in our model we would like to include an additional term reflecting possible instability of the client's mood due to mental comorbidity, which we reflect by the function $I_C(C)$ yielding

$$\dot{C} = -m_C C + b_C + I_C(C) \tag{1}$$

For this reason, in the following we will mainly focus on the function I_C being periodic, because it will allow us to better reflect mental comorbidity implying changes occurring in the client's emotional states. This means that instead of having one favorable state, the client can achieve many different states, depending on the model parameters and initial data.

Note that the equation m_CC - b_C = I_C(C) has at least one solution C^* (Fig. 1). The equilibrium C^* of Eq. 1 is stable, if $I_C'(C^*) < m_C$. It is also worth noting that from the point of view of applications, in a situation where there are many equilibrium points, we are only interested in cases with the smallest and largest equilibria being stable. Otherwise, clients' emotions may get out of control and go (in the model) to infinity, which is not possible in reality. Additionally, by identifying their extreme emotional states, we can better adjust the therapy process.

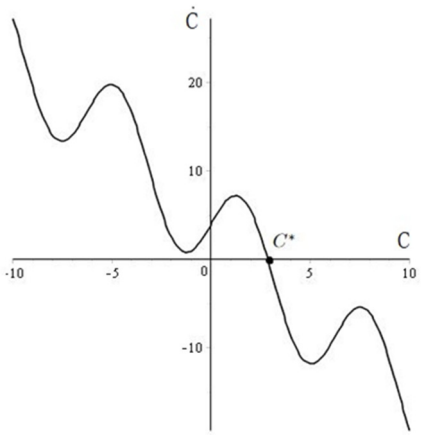


Fig. 1. Possible equilibria of Eq. 1 – solutions to $m_C C - b_C = I_C(C)$.

Moreover, let us assume that the state of the client is also influenced by the external environment, understood here as a group of people with whom the client is in some relationship. Other environmental factors such as housing and food insecurity, racist and sexist institutions, quantity and quality of free time, working conditions etc. can be included in the model and reflected by an additional non-positive constant e_C in the equation describing the client's emotional state. In the following we assume that $e_C = 0$, which does not affect the qualitative analysis of the model. Moreover, we assume that this group is large enough that the impact of one person on the state of this group is negligible, if all of the people in this group impact similarly on the client. This is assumed for simplicity, while in general this impact should be also taken into account, for example in the context of family relationships. In particular, if family members exert a more abusive and destructive influence on the client compared to friends or work colleagues, then the influence of the latter is negligible. Denoting by $E(t)$

the state of the external environment at time t, influencing the client's mood $C(t)$ and considering Eq. 1, we obtain the following system of differential equations

$$\begin{aligned} \dot{C} &= -m_C C + b_C + I_C(C, E), \\ \dot{\dot{E}} &= -m_E E + b_E. \end{aligned} \tag{1}$$

In System 2, I_C is a certain function describing the impact of the client's mood and the environment on the client, which will be defined later, while $m_E > 0$ and b_E describe the external environment. In this paper, we assume that the social environment has a simple dynamic, reflected by a linear right-hand side in System 2 and that it affects the client in a pathological manner, that is its impact is negative.

The goal of psychotherapy is to help the client manage their issues or remedy them entirely. The therapeutic success may be measured by how well this task is carried out. According to Peluso et al. (2012), between the client and their therapist there is a "usual" human relationship, and therefore during the therapeutic session the relation between them can be described as a dyadic interaction as in article Liebovitch et al. (2011) as follows

$$\begin{aligned} \dot{T} &= I_T(C, T), \\ \dot{C} &= -m_C C + b_C + I_C(C, T). \end{aligned}$$

where $T(t)$ reflects the therapist's state during the therapeutic session, while I_T and I_C are the functions describing dyadic interactions between the therapist and his/her client and vice versa, respectively. In our article, the state T of the therapist could be also meant as the level of therapeutic quality during the session with the client at time t.

It seems that during the therapeutic session the influence of social environment is not so important, while after the session the influence of therapist is much smaller than the influence of the social environment, which means that we should consider switches between two systems describing these two situations. However, as a first approximation to this problem we would like to propose a system of equations reflecting both: the influence of the social environment and the therapist at the same time. Taking into account the above considerations, the full system describing psychotherapy takes the form

$$\begin{aligned} \dot{T} &= I_T(C, T), \\ \dot{C} &= -m_C C + b_C + I_C(C, T, E), \\ \dot{\dot{E}} &= -m_E E + b_E. \end{aligned} \tag{2}$$

In System 3, the state of the client should remain unchanged under unchanged conditions, i.e., if neither variation in the trait parameters of the client nor any

input from the therapist or from other social environments are introduced. The same reasoning can be applied to the therapist state. Therefore, we will consider the following influence functions

$$I_T(C,T) \coloneqq I_T(C) + I_T(T)$$

and

$$I_C(C,T,E) \coloneqq I_C(C) + I_C(T) + I_C(E).$$

Consequently, System 3 takes the following form

$$\begin{aligned} \dot{T} &= I_T(C) + I_T(T), \\ \dot{C} &= -m_C C + b_C + I_C(C) + I_C(T) + I_C(E), \\ \dot{\dot{E}} &= -m_E E + b_E. \end{aligned} \tag{3}$$

In the following we will focus on the dynamics of System 4 with the simplest linear function $I_T(T) \coloneqq -m_T(T - k_T)$, $m_T > 0$, which means that in the solitude, the therapist has a neutral state, while $k_T \geqslant 0$ reflects the competences and skills of the therapist in the context of the psychotherapy of the specific client, such as: selection of therapeutic techniques to the client's condition, empathy, commitment, caring etc.

Let us introduce the notation

$$I_T(C) \coloneqq c_1 f_1(C),$$

$$I_C(T) \coloneqq c_2 f_2(T),$$

$$I_C(E) \coloneqq c_3 f_3(E),$$

$$I_C(C) \coloneqq c_4 f_4(C).$$

Then System 4 takes the following form

$$\begin{aligned} \dot{T} &= -m_T(T - k_T) + c_1 f_1(C), \\ \dot{C} &= -m_C C + b_C + c_2 f_2(T) + c_3 f_3(E) + c_4 f_4(C) \\ \dot{\dot{E}} &= -m_E E + b_E. \end{aligned} \tag{4}$$

where m_T, m_C, $m_E > 0$ and other model parameters can have different signs.

Throughout the paper, we will make use of the following assumptions:

(F1) $f_i \in \boldsymbol{C}^2(\mathbb{R}), i = 1, \dots, 4$

(F2) $f_i(0) = 0, i = 1, \dots, 4$

(F3) $f_i'(x) > 0$ for $x \in \mathbb{R}, i = 1,2,3;$

(F4) $xf_i''(x) \leqslant 0$ for $x \in \mathbb{R}, i = 1,2,3;$

(F5) f_4 is periodic.

Now, we would like to point out the differences between our approach and the model from Liebivitch et al. (2011). Firstly, System 5 is 3-dimensional, as it takes into account the state of the external environment, which was not considered in Liebovitch et al. (2011). Note that, while components of the equations describing the emotional states of each relevant group in solitude were inspired by Liebovitch's article, the influence functions were defined in a broader manner, as presented in the above assumptions. An additional difference is the use of the periodic function to describe the client's emotional instability.

BASIC PROPERTIES

Assumption (F1) guarantees existence and uniqueness of local solutions to System 5. Moreover, from Assumptions (F1) - (F4), f_i, i = 1, 2, 3, has sublinear growth (is linearly bounded). This implies that the right-hand sides of System 5 have sublinear growth. Consequently, for any initial condition there exists exactly one global solution to System 5 (Amann, 1990). Under some additional assumptions solutions to System 5 are also bounded.

Theorem 1

Let Assumptions (F1) - (F5) hold and the function f_1 be bounded. Then solutions to System 5 are bounded for $t \geqslant 0$.

Proof

Boundedness is obvious for the last variable E. Clearly, $|E(t)| \leqslant |b_E \backslash m_E| + |E(0)|$ for all $t \geqslant 0$.

Let us consider the first variable T. If f_1 is bounded, then there exists $A_1 > 0$ such that $|c_1 f_1(C)| \leqslant A_1$ for any $C \in R$. Consequently,

$$-m_T(T - k_T) - A_1 \leqslant \dot{T} \leqslant -m_T(T - k_T) + A_1.$$

Using special case of the Gronwall's inequality, i.e. if $u' \leqslant -au + b$ then $u(t) \leqslant u(0)\exp(-at) + b / a(1 - \exp(-at))$ we get

$$T(t) \leqslant \left(k_T + \frac{A_1}{m_T}\right)(1 - e^{-m_T t}) + T(0)e^{-m_T t}.$$

Similarly, to estimate the variable T from below, we use the Gronwall's inequality, i.e. if $v' \leqslant -av + b$ then $v(t) \leqslant v(0)\exp(-at) + b / a(1 - \exp(-at))$, where $v = -u$ and we get

$$T(t) \geqslant \left(k_T - \frac{A_1}{m_T}\right)(1 - e^{-m_T t}) + T(0)e^{-m_T t}.$$

Hence

$$|T(t)| \leqslant k_T + \frac{A_1}{m_T} + |T(0)| \text{ for } t \geqslant 0.$$

Having bounded T and E, and due to the periodicity of f_4, there exists $A_2 > 0$ such that

$$|b_C + c_2 f_2(T) + c_3 f_3(E) + c_4 f_4(C)| \leqslant A_2$$

for any solution (T, C, E) to System 5. Hence, using the same arguments as above, one gets

$$|C(t)| \leqslant \frac{A_2}{m_C} + |C(0)| \text{ for } t \geqslant 0$$

Proceeding similarly to the proof of Theorem 1, one can prove the following theorem.

Theorem 2

If Assumptions (F1) - (F5) are satisfied and f_2 is bounded, then solutions to System 5 are bounded for $t \geqslant 0$.

Remark 1

Note that if f_1 and f_2 are unbounded this does not necessarily mean that solutions are unbounded as well.

Indeed, let $f_1(C) = C$ and $f_2(T) = T$ and consider the following special case of the model

$$\dot{T} = -4T + 2C,$$

$$\dot{C} = -3C + T,$$

for $T(0) = T_0$ and $C(0) = C_0$, where $T_0 = C_0$. Then we have

$$T(t) = (2C_0 - T_0)e^{-2t} + 2(T_0 - C_0)e^{-5t} \leqslant T_0, \text{ and}$$

$$C(t) = (2C_0 - T_0)e^{-2t} + (T_0 - C_0)e^{-5t} \leqslant C_0.$$

Theorem 3

Let Assumptions (F1) - (F5) hold and at least one of the parameters c_i, $i = 1, 2$, be non-negative. If for the solution (T(t), C(t), E(t)) to System 5 there exists $t_0 \geqslant 0$ such that (T(t), C(t)) remains in either second or fourth quadrant of $\mathbb{R}^2$ for all $t > t_0$, then this solution is bounded for $t \geqslant 0$.

Proof

Since E is bounded, it is sufficient to show boundedness of T and C. We consider two cases.

Case 1. Assume that ($T(t)$, $C(t)$) lies in the second quadrant for all $t > t_0$, i.e., $T(t) \leqslant 0$, $C(t) \geqslant 0$ for all $t > t_0$.

Then for $c_1 \geqslant 0$ and $c_2 \in \mathbb{R}$ we have $c_1 f_1(C(t)) \geqslant 0$ for $t > t_0$, due to (F2), (F3). Consequently, we have

$$\dot{T} \geqslant -m_T(T - k_T) \geqslant 0,$$

which leads to $T(t) \geqslant T(t_0)$ for $t > t_0$. Thus, we conclude that $T(t) \in [T(t_0), 0]$ for $t > t_0$. It remains to prove that $C(t)$ is bounded from above. We have

$$\leqslant -m_C C + b_C + |c_2| \max_T |f_2(T)| + |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)|,$$

$$\leqslant -m_C C + A,$$

where

$$A := b_C + |c_2| \max_T |f_2(T)| + |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)|,$$

Due to Gronwall's inequality we have

$$C(t) \leqslant C(t_0) + \frac{b_C + A}{m_C} \text{ for } t \geqslant 0.$$

If $c_2 \geqslant 0$ and $c_1 \leqslant 0$ we proceed analogously. First, we show that $C(t)$ is bounded from above. We have

$$\dot{C} \leqslant -m_C C + b_C + c_2 f_2(T) + |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)| \leqslant -m_C C + \bar{A}$$

where

$$\bar{A} := b_C + |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)|.$$

Again, by Gronwall's inequality we get

$$C(t) \leqslant C(t_0) + \frac{b_C + \bar{A}}{m_C} =: C_{max} \text{ for } t \geqslant 0.$$

Now, it remains to prove that T is bounded from below for $t > t_0$. Due to (F3) we have

$$\dot{T} \geqslant -m_T T + c_1 f_1(C_{max}),$$

So

$$T(t) \geqslant T(t_0) + \frac{c_1 f_1(C_{max})}{m_T} for\ t \geqslant 0.$$

Case 2. Assume that $(T(t), C(t))$ lies in the fourth quadrant for all $t > t_0$, i.e., $T(t) \geqslant 0$ and $C(t) \leqslant 0$.

If $c_1 \geqslant 0$ and $c_2 \in \mathbb{R}$, then due to (F2) and (F3) we have $c_1 f_1(C(t)) \leqslant 0$ for $t > t_0$ and we obtain

$$T(t) \leqslant T(t_0) + k_T \text{ for } t \geqslant 0.$$

which leads (yet again by Gronwall's inequality) to

$$T(t) \leqslant T(t_0) + k_T \text{ for } t > t_0.$$

proving that T is bounded from above. It remains to prove that C is bounded from below. We have

$$\dot{C} \geqslant -m_C C + b_C - |c_2| \max_T |f_2(T)| - |c_3| \max_E |f_3(E)| - |c_4| \max_C |f_4(C)| \geqslant -m_C C + |D|$$

where

$$D := b_C - |c_2| \max_T |f_2(T)| + |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)|.$$

This implies that

$$C(t) \geqslant C(t_0) - \frac{|D|}{m_C}.$$

meaning that C is bounded from below.

The case when $c_1 \leqslant 0$ and $c_2 \geqslant 0$ implies that $c_2 f_2(T(t)) \geqslant 0$ for $t > t_0$. We first show that $C(t)$ is bounded from below. We have

$$\begin{aligned} &\geqslant -m_C C + b_C + c_2 f_2(T) - |c_3| \max_E |f_3(E)| - |c_4| \max_C |f_4(C)| \\ &\geqslant -m_C C + b_C + c_2 f_2(T) - \overline{D} \\ &\geqslant -m_C C \ |\overline{D}| \end{aligned}$$

where

$$\overline{D} := b_C - |c_3| \max_E |f_3(E)| + |c_4| \max_C |f_4(C)|.$$

It remains to prove that T is bounded from above. We have

$$\dot{T} \leqslant -m_T(T - k_T) + c_1 f_1(C) \leqslant -m_T(T - k_T) + c_1 f_1(C_{\max}).$$

which implies

$$T(t) \leqslant T(t_0) + k_T + \frac{c_1 f_1(C_{\max})}{m_T} \text{ for } t > t_0.$$

Hence, we get that T is bounded from above.

Theorem 4

Let Assumptions (F1) - (F5) hold. Moreover, assume that

$$m_T > |c_2| f_2'(0) \text{ and } m_C > |c_1| f_1'(0). \tag{6}$$

If for the solution (T(t),C(t), E(t)) to Sysyem5 there exists $t \geqslant 0$ such that (T(t),C(t)) remains in either first or third quadrant of $\mathbb{R}^2$ for all $t > t_0$, then this solution is bounded for $t \geqslant 0$.

Proof

As in the proof of Theorem 3, E is bounded and we need to show that T and C are also bounded.

First, assume that $(T(t), C(t))$ lies in the first quadrant for all $t > t_0$. Then T and C are bounded from below by 0. We shall show that $T(t) + C(t)$ is bounded from above. Due to (F2), (F4) and Mean Value Theorem, for $x \geqslant 0$ we have $f_i(x) \leqslant f_i'(0)x$, $i = 1, 2$. Hence, we get

$$\begin{aligned}\frac{d}{dt}(T+C) &\leqslant -m_T T + |c_1| f_1'(0) C - m_C C + |c_2| f_2'(0) T + A_1 \\ &= -\big(m_T - |c_2| f_2'(0)\big) T - \big(m_C - |c_1| f_1'(0)\big) C + A_1.\end{aligned}$$

where

$$A_1 = m_T k_T + |b_C| + \max_E |c_3||f_3(E)| + \max_C |c_4||f_4(C)| < \infty.$$

From Inequalities 6, due to the periodicity of f_4, continuity of f_3 and boundedness of E, we obtain

$$\frac{d}{dt}(T+C) \leqslant -\min\{m_T - |c_2| f_2'(0), m_C - |c_1| f_1'(0)\}(T+C) + A_1.$$

Consequently, for all t≥t_0 , we obtain by Gronwall's inequality¨

$$T(t) + C(t) \leq T(t_0) + C(t_0) + \frac{A_1}{\min\{m_T - |c_2| f_2'(0), m_C - |c_1| f_1'(0)\}}$$

Hence, both T and C are bounded.

Now, assume that (T(t), C(t)) lies in the third quadrant for all $t > t_0$Then T and C are bounded from above by 0. We shall show that $T(t) + C(t)$ is bounded from below. Due to (F2), (F4) and Mean Value Theorem, for $x \leqslant 0$ we have $f_i(x) \geqslant f_i'(0)x$, $i = 1, 2$. Hence, we get

$$\begin{aligned}\frac{d}{dt}(T+C) &\leqslant -m_T T + |c_1| f_1'(0) C - m_C C + |c_2| f_2'(0) T + A_2 \\ &= -\big(m_T - |c_2| f_2'(0)\big) T - \big(m_C - |c_1| f_1'(0)\big) C + A_2 \\ &\geqslant \min\big\{m_T - |c_2| f_2'(0), m_C - |c_1| f_1'(0)\big\}(T+C) + A_2,\end{aligned}$$

where

$$A_2 = m_T k_T + |b_C| - |c_3| \max_E |f_3(E)| - |c_4| \max_C |f_4(C)| < \infty,$$

which implies that

$$T(t) + C(t) \geqslant T(t_0) + C(t_0) - \frac{|A_2|}{\min\{m_T - |c_2| f_2'(0), m_C - |c_1| f_1'(0)\}}$$

Consequently, T and C are bounded.

EXISTENCE OF EQUILIBRIA

Equilibria of System 5 can be obtained by solving the following algebraic equations

$$0 = -m_T(T - k_T) + c_1 f_1(C),$$
$$0 = -m_C C + b_C + c_2 f_2(T) + c_3 f_3(E) + c_4 f_4(C) \quad (7)$$
$$0 - m_E E + b_E.$$

Then $E := E^* = b_E / m_E$ and one gets

$$0 = -m_T(T - k_T) + c_1 f_1(C),$$
$$0 = -m_C C + b_C + c_2 f_2(T) + c_3 f_3(E) + c_4 f_4(C) \quad (8)$$

Set

$$a_i^{\pm} := \lim_{x \to \pm\infty} f_i(x)\,, i = 1, 2, \quad (9)$$

and define

$$\mathcal{L} := \{(C, T): -m_T(T - k_T) + c_1 f_1(C) = 0\},$$
$$\mathcal{M} := \{(C, T): -m_C C + b_C + c_2 f_2(T) + c_3 f_3(E^*) + c_4 f_4(C) = 0\}.$$

For the curve $\mathcal{L}$, let us consider

$$T_{\mathcal{L}}(C) = \frac{c_1 f_1(C)}{m_T} + k_T,$$

and therefore

$$T'_{\mathcal{L}}(C) = \frac{c_1 f'_1(C)}{m_T},$$

Since $m_T > 0$, from Assumption (F3), we conclude that $T_{\mathcal{L}}$is increasing if $c_1 > 0$ and decreasing when $c_1 < 0$. Moreover,

$$T''_{\mathcal{L}}(C) = \frac{c_1 f''_1(C)}{m_T}.$$

Firstly, let the inequality in Assumption (F4) be strict for $i = 1$ and $C \neq 0$. Then $T_{\mathcal{L}}$ is concave for $C > 0$ and convex for $C<0$, if $c_1>0$ and $T_{\mathcal{L}}$ is concave for $C < 0$ and convex for $C > 0$, if $c_1 < 0$. Note that, since Assumption (F3) holds, if $f''_1(C) = 0$ in (F4) for all $C \in \mathbb{R}$, then $T_{\mathcal{L}}$ is a linear function. Exemplary graphs of the curve $\mathcal{L}$ are shown in Fig. 2.

Now, let us consider the curve M which we treat as a graph of $T_M(C)$. From the Implicit Function Theorem, one has

$$T''_{\mathcal{L}}(C) = \frac{c_1 f''_1(C)}{m_T}. \quad (10)$$

and the other has

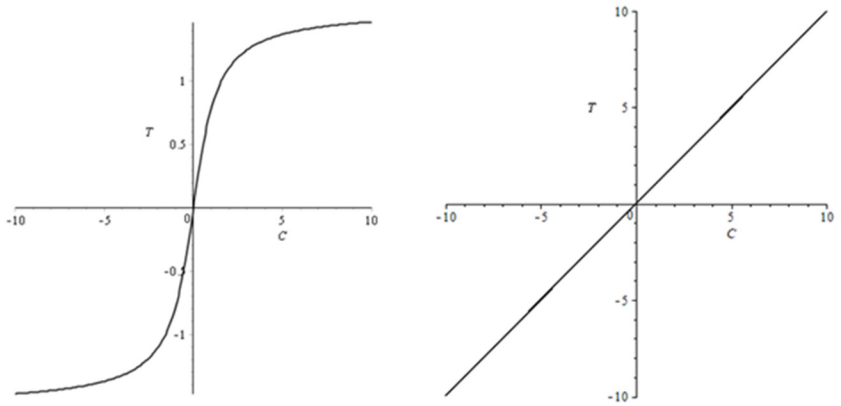


Fig. 2. Exemplary graphs of the curve $\mathcal{L}$ for (a) $a_1^{\pm} \in \mathbb{R}$ and (b) $a_1^{\pm} = \pm\infty$.

$$T''_{\mathcal{M}}(C) = \frac{-c_4 f_4''(C)(c_2 f_2'(T))^2 - c_2 f_2''(T)(m_C - c_4 f_4'(C))^2}{(c_2 f_2'(T))^3}. \tag{11}$$

The domain of the function T_M depends on the values of the function f_2. We denote the domain of T_M by D. It is easy $|a_2^{\pm}| < \infty$ (f_2 is bounded) then D is bounded and $T_M(C)$ tends to $\pm\infty$ when C converges to the boundary to see that if a_2 of D.

Note that the function

$$F(C,T) = -m_C C + b_C + c_2 f_2(T) + c_3 f_3(E^*) + c_4 f_4(C)$$

is of class $\boldsymbol{C}^2$, the function $T \to F(C, T)$ is continuous and if $c_2 > 0$, then $F'_T(C, T) > 0$, and when $c_2 > 0$, then $F'_T(C, T) < 0$ for every fixed $C \in D$. Thus, from the Darboux property and strict monotonicity, for every $C \in D$, if there is $T \in \mathbb{R}$ such that $F(C, T) = 0$, then T is unique. Consequently, there is exactly one implicit mapping $T = T(C)$ and it is continuous in D.

The difference of the slopes of the curves $\mathcal{L}$ and $\mathcal{M}$ will also be useful in further considerations. We have

$$T'_{\mathcal{L}-\mathcal{M}}(C) = \frac{c_1 c_2 f_1'(T) f_2'(C) - m_T m_C + m_T c_4 f_4'(C)}{m_T c_2 f_2'(T)}. \tag{12}$$

We shall divide the analysis of the existence of equilibria into two cases.

Case 1: $c_4 = 0$ (the client is stable)

Then, from Eq. 10 and since $m_C > 0$, T_M is increasing if $c_2 > 0$ and decreasing form $c_2 < 0$. Moreover, by Eq. 11,

$$T''_{\mathcal{M}}(C) = \frac{-m_C^2 f_2''(T)}{c_2^2 f_2'(T)^3}.$$

In the case when in Assumption (F4) the inequality is strict for $T \neq 0$, T_M is convex for $T > 0$ and concave for $T < 0$. If $f_2''(T) = 0$ in Assumption (F4) for all $T \in \mathbb{R}$, by (F3), T_M is linear. Exemplary graphs of the curve M are shown in Fig. 3. We shall now consider four subcases.

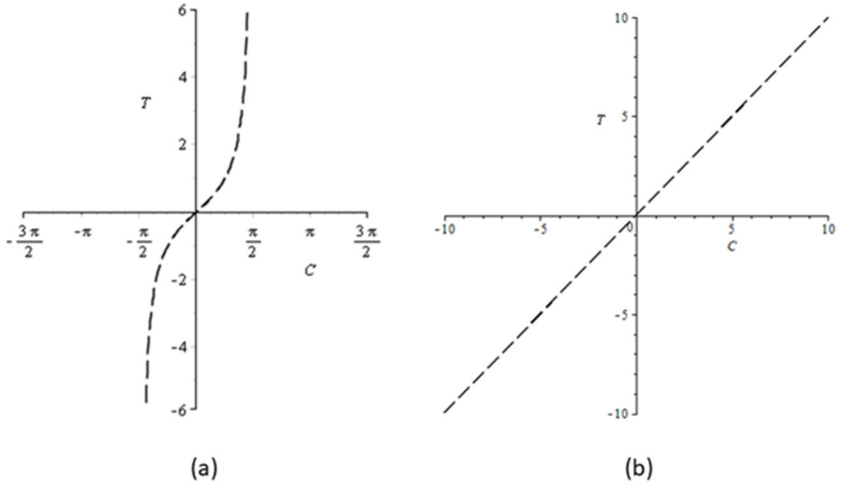


Fig. 3. Exemplary graphs of the curve $\mathcal{M}$ for (a) $a_2^{\pm} \in \mathbb{R}$ and (b) $a_2^{\pm} = \pm\infty$.

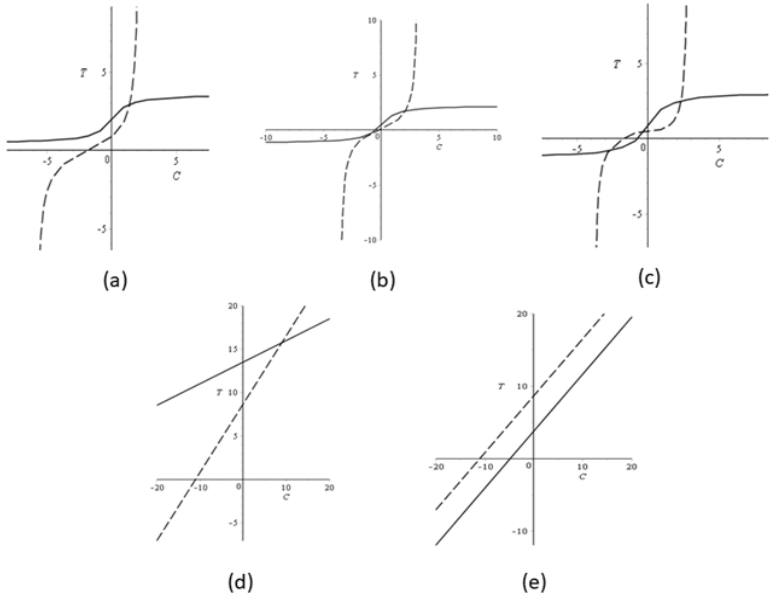


Fig. 4. Case 1A - the points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$.

Case 1A: $c_1, c_2 > 0$

In this case $\mathcal{L}$ and $\mathcal{M}$ are increasing and the curves have from 0 to 3 points of intersection. Indeed, taking into account the previous considerations and the shapes of the curves $\mathcal{L}$ and $\mathcal{M}$ resulting from the assumptions posed, all possible intersections of the curves are shown in Fig. 4. Let $a_j^{\pm} \in \mathbb{R}$, $j = 1, 2$ as defined in Eq. 9. In the case when (C_1, T_1) is the only one intersection point of the curves $\mathcal{L}$ and $\mathcal{M}$, we have $T_l'(C_1) < T_m'(C_1)$, i.e., $T'_{\mathcal{L}-\mathcal{M}}(C_1) < 0$as defined in Eq. 12, which implies that $c_1c_2f_1'(T_1)f_2'(C_1) - m_Tm_C < 0$. Consequently, if ($C1$, $T1$) is a solution to System 8 and $c_1c_2f_1'(T_1)f_2'(C_1) - m_Tm_C < 0$and there is no solution to System 8 satisfying $c_1c_2f_1'(T_2)f_2'(C_2) - m_Tm_C \geqslant 0$, then System 8 has one solution as shown in Fig. 4(a) and (d). If System 8 has two solutions: (C_1, T_1) and (C_2, T_2), $C_1 < C_2$, then, by Eq. 11 $c_1c_2f_1'(C_2)f_2'(T_2) - m_Tm_C < 0$ and $c_1c_2f_1'(T_1)f_2'(C_1) - m_Tm_C = 0$ as shown in Fig. 4 (b). If there are three solutions to System 8: (C_1, T_1)(C_2, T_2) and (C_3, T_3) with $C_1 < C_2 < C_3$, then $T'_{\mathcal{L}-\mathcal{M}}(C_1) < 0$ and $T'_{\mathcal{L}-\mathcal{M}}(C_3) > 0$ what is presented in Fig. 4(c). Moreover, if $a_i^{\pm} = \pm\infty$, $i = 1, 2$, $T_{\mathcal{L}}(0) \neq T_{\mathcal{M}}(0)$ and $T'_{\mathcal{L}-\mathcal{M}}(C) = 0$, then System 8 has no solution as shown in Fig. 4(e).

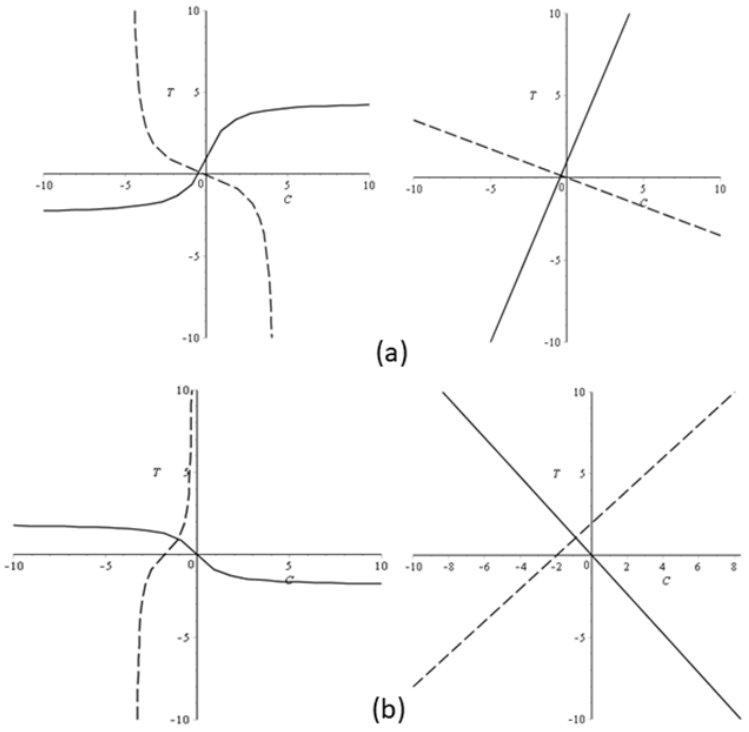


Fig. 5. The points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$ (a) Case 1C (b) Case 1D.

Case 1B: $c_1, c_2 < 0$

This case is symmetric to Case 1A. Now, $\mathcal{L}$ and $\mathcal{M}$ are decreasing. First suppose that $\mathcal{L}$ and $\mathcal{M}$ have one intersection point (C_1, T_1). Then, using the previous considerations on the shape of the curves $\mathcal{L}$ and $\mathcal{M}$, we have $T'_{\mathcal{L}-\mathcal{M}}(C_1) > 0$. However, now c_2 is negative and Eq. 12 implies that $c_1 c_2 f_1'(T_1) f_2'(C_1) - m_T m_C < 0$. So the inequality has the same sign as in Case 1A. Further analysis of the existence of solutions to System 8 in this case is analogous to that in Case 1A.

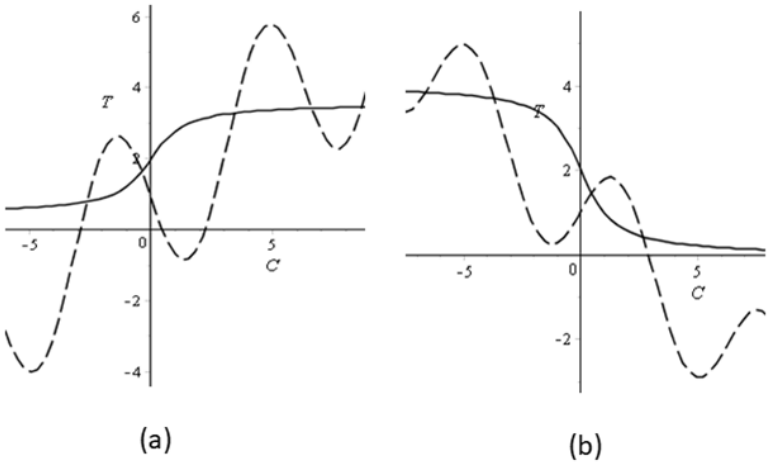


Fig. 6. The points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$ for Case 2: (a) $c_1, c_2 > 0$ or (b) $c_1, c_2 < 0$

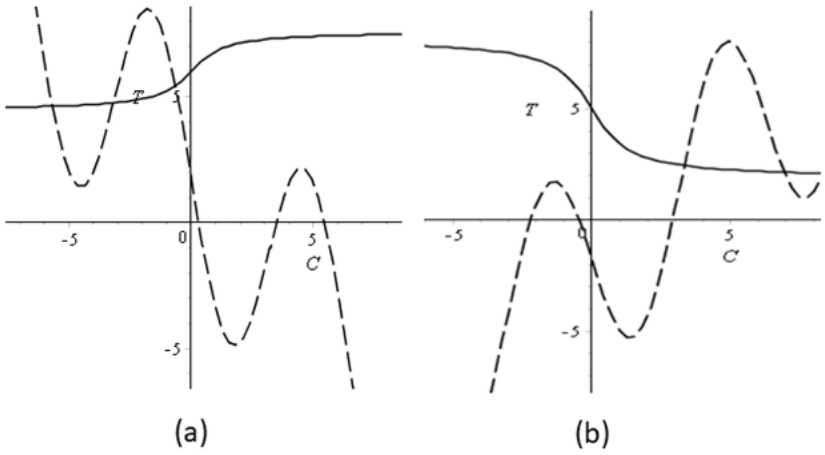


Fig. 7. The points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$ for Case 2: (a) $c_1 > 0, c_2 <$ or (b) $c_1 < 0, c_2 > 0$

Case 1C: $c_1 > 0$, $c_2 < 0$

In this case $\mathcal{L}$ is increasing and $\mathcal{M}$ is decreasing. Since $\mathcal{L}$ is defined on $\mathbb{R}$ and T_M (C) $\to \pm\infty$ when C converges t the boundary of its domain (which can be bounded), there is always one intersection point as shown in Fig. 5(a).

Case 1D: $c_1 < 0$, $c_2 > 0$

Now, $\mathcal{L}$ is decreasing and $\mathcal{M}$ is increasing. The curves have one intersection point as shown in Fig. 5(b).

Case 2: $c_4 \neq 0$

Let us first assume that the functions f_1, f_2 are bounded $a_j^{\pm} \in \mathbb{R}$, $j = 1, 2$. Then, in the formulas for the curves $\mathcal{L}$ and $\mathcal{M}$, the functions f_1, f_2 are dominated by the linear terms: $-m_T T$ and $-m_C C (m_C, m_T > 0)$. Consequently, Case 2 is no different from Case 1 and the points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$ are like in Figs. 4-5.

If we assume that $a_1^{\pm} \in \mathbb{R}$ (f_1 is bounded) and then we get finitely many points of intersection of the curves $\mathcal{L}$ and $\mathcal{M}$ as shown in Fig. 6 and Fig.7.

We can obtain infinitely many solutions to System 8, if we assume that $a_2^{\pm} = \pm\infty$, $j = 1, 2$, and when $\mathcal{L}$ is included in the graph $\mathcal{M}$ in the sense of Fig. 8. Note that it is possible only in the cases when $c_1, c_2 > 0$ or $c_1, c_2 < 0$. It is easy to see that we get infinitely many points of intersection also when case $\mathcal{L}$ is included in the graph $\mathcal{M}$ for $C \in (a, \infty)$ or $C \in (-\infty, b)$, $a, b \in \mathbb{R}$.

Note that, by Eq. 12, the sign of c_2 and

$$c_1 c_2 f_1'(T) f_2'(C) - m_T m_C + m_T c_4 f_4'(C)$$

play the most important role in Case 2.

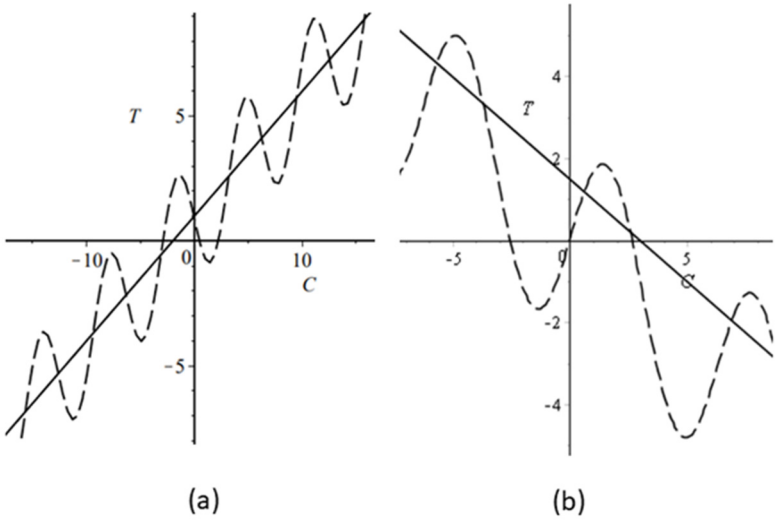


Fig. 8. Infinitely many intersection points of the curves $\mathcal{L}$ and $\mathcal{M}$ in Case 2.

STABILITY OF THE EQUILIBRIA

Studying local stability, we consider Jacobian matrix of System 5. This matrix reads

$$J(T,C,E)=\begin{pmatrix} -m_T & c_1 f_1'(C) & 0 \\ c_2 f_2'(T) & -m_C + c_4 f_4'(C) & c_3 f_3'(E) \\ 0 & 0 & -m_E \end{pmatrix}.$$

Let $P_i := (T_i, C_i, E_i)$, with $i \in \mathbb{N}$ fixed, be an equilibrium. The local stability of P_i is determined by eigenvalues of the matrix $J(P_i)$. We have

$$\det(J(P_i)-\lambda\mathbb{I}) = \begin{vmatrix} -m_T-\lambda & c_1 f_1'(C_i) & 0 \\ c_2 f_2'(T_i) & (-m_C + c_4 f_4'(C_i)) - \lambda & c_3 f_3'(E_i) \\ 0 & 0 & -m_E-\lambda \end{vmatrix}$$

$$= -(\lambda^2 + (m_T + m_C - c_4 f_4'(C_i))\lambda + m_T m_C - m_T c_4 f_4'(C_i) - c_1 c_2 f_1'(C_i) f_2'(T_i))(m_E + \lambda)$$

Clearly, $\lambda_3 = -m_E$ and eigenvalues λ_1 and λ_2 satisfy the quadratic equation

$$\lambda^2 + (m_T + m_C - c_4 f_4'(C_i))\lambda + m_T m_C - m_T c_4 f_4'(C_i) - c_1 c_2 f_1'(C_i) f_2'(T_i) = 0.$$

Set

$$\Delta := (m_T + m_C - c_4 f_4'(C_i))^2 + 4(c_1 c_2 f_1'(C_i) f_2'(T_i) - m_T m_C + m_T c_4 f_4'(C_i)) \quad (13)$$

According to the results obtained in Section 3, we will divide the stability analysis into cases.

Case 1

In this case we have

$$\lambda^2 + (m_T + m_C)\lambda + m_T m_C - c_1 c_2 f_1'(C_i) f_2'(T_i) = 0,$$

$$\Delta = (m_T + m_C)^2 + 4(c_1 c_2 f_1'(C_i) f_2'(T_i) - m_T m_C).$$

Note that

$$\lambda_2 + \lambda_3 = -(m_T + m_C) < 0. \quad (14)$$

Moreover, if $\Delta < 0$, then

$$\lambda_{2,3} = \frac{1}{2}\left(-(m_T + m_C) \pm \sqrt{-\Delta}i\right).$$

Lemma 1

Let Case 1C or 1D hold. Then the unique equilibrium point P_1 is locally asymptotically stable as a solution to System 5.

Proof

If c_1, $c_2 < 0$ (Case 1C, 1D), then, for $\Delta \geqslant 0$,

$$\lambda_2\lambda_3 = -(c_1c_2f_1'(C_1)f_2'(T_1) - m_Tm_C) > 0 \tag{15}$$

as defined in Assumption (F3). Hence, from Eq. 14, λ_2, $\lambda_3 < 0$. In the case when $\Delta < 0$,

$$\mathrm{Re}\lambda_{2,3} = \frac{1}{2}(-m_T + m_C) < 0. \tag{16}$$

Consequently, P_1 is stable.

Further lemmas will apply to Case 1A. In this case we can have from 0 to 3 equilibrium points.

Lemma 2

If for some $I \in \{1, 2, 3\}$ the following inequality.

$$c_1c_2f_1'(C_i)f_2'(T_i) - m_Tm_C < 0 \tag{17}$$

is satisfied, then the equilibrium $P_i := (T_i, C_i, E_i)$, of System 5 is stable.

Assume that for an $i \in \{1, 2, 3\}$ we have

$$c_1c_2f_1'(C_i)f_2'(T_i) - m_Tm_C < 0 \tag{18}$$

then P_i is unstable.

Proof

Note that if Inequality 17 is satisfied, then, by Eq. 13, Eq. 14 and Eq. 16 either $\lambda_i < 0$, $i = 1, 2, 3$, or $\lambda_1 < 0$ and $\mathrm{Re}\lambda_{2,3} < 0$. If Inequality 18 holds, then $\Delta > 0$ and, by Eq. 15, λ_2, $\lambda_3 < 0$. Consequently, P_i is unstable.

First, consider Case 1A with only one equilibrium P_i what is shown in Fig. 4(a). From the considerations in Existence of the equilibria, Eq. 12 is negative. Since in this case $c_2 > 0$, Inequality 17 holds, which implies that P_1 is (locally) asymptotically stable.

Let Case 1A with two intersection points hold what is shown in Fig. 4(b). If (C_1, T_1) is the point where the curves $\mathcal{L}$ and $\mathcal{M}$ meet but do not intersect, then

$$c_1c_2f_1'(C_1)f_2'(T_1) - m_Tm_C = 0.$$

Hence, by Eq. 13, $P_1 = (T_1, C_1, E_1)$ is not hyperbolic. For the second equilibrium $P_2 = (T_2, C_2, E_2)$, Inequality 17 is satisfied. Consequently, P_2 is stable.

If we consider Case 1A with three equilibrium points: P_1, P_2, P_3 with $C_1 < C_2 < C_3$, then it is easy to see that

$$c_1c_2f_1'(C_2)f_2'(T_2) - m_Tm_C > 0$$

and

$$c_1c_2f_1'(C_i)f_2'(T_i) - m_Tm_C < 0,$$

where $i \in \{1, 3\}$. Consequently, P_2 is unstable and P_1, P_3 are stable.

Case 1B is symmetric to Case 1A what is shown in Figs. 4. Considering the shape of the curves $\mathcal{L}$ and $\mathcal{M}$, we obtain that, in the case of one intersection point, Eq. 12 is positive. But, in Case 1B, $c_2 > 0$ and therefore Inequality 16 is satisfied. Consequently, Lemma 6 holds true also for Case 1B. For Case 1B with three equilibrium points: P_1, P_2, P_3 with $C_1 < C_2 < C_3$, P_2 is unstable and P_1, P_3 are stable.

For Case 1A-B we know that we can get three equilibrium point, where two of them are asymptotically stable. The equilibrium point in this case will be globally stable if it is unique. We formulate appropriate theorem:

Theorem 5

Let $c_1c_2 > 0$, (T^*, C^*, E^*) be a solution to System 7 i.e., it is an equilibrium point of System 5. Then System

$$\begin{aligned} u' &= -m_T u + c_1(f_1(v + C^*) - f_1(C^*)) \\ v' &= -m_C v + c_2\left(f_2(u + T^*) - f_2(T^*)\right) + c_3\left(f_3(w + E^*) - f_3(E^*)\right) + c_4(f_4(v + C^*) - f_4(C^*)) \\ w'' &= -m_E w. \end{aligned} \tag{19}$$

has an equilibrium point (0, 0, 0). Moreover, this equilibrium point is globally stable if

1. $f_1'(v)f_2'(u) < \frac{m_T m_C}{c_1 c_2}$ for all $u, v \in \mathbb{R}$,
2. (T^*, C^*, E^*) *is a unique equilibrium point of System* 5.

Proof

We start with changing variables of System 5 such that the equilibrium point (T^*, C^*, E^*) is shifted to (0, 0, 0). We define $\underline{u} = T - T^*$, $\underline{v} = C - C^*$ and $\underline{w} = E - E^*$. It is obvious that (0, 0, 0) is a equilibrium point of System 19. We focus on proving global stability of this equilibrium. Let us define:

$$L(u, v, w) = |c_2| \int_0^u (f_2(\zeta + T^*) - f_2(T_*))d\zeta + |c_1| \int_0^v (f_1(\zeta + C^*) - f_1(C_*))d\zeta + A\frac{w^2}{2} \tag{20}$$

where

$$A = \frac{m_T c_3^2 |c_1| f_1'(0) f_3'(0)^2}{4m_E(m_T m_C - c_1 c_2 f_1'(0) f_1'(0) f_2'(0))} + 1.$$

It is obvious that $L(u, v, w) = 0$ iff $u = v = w = 0$. We use the Mean Value Theorem to calculate the derivative of $\mathcal{L}$ along trajectories of System 5:

$$\begin{aligned} L'(u, v, w) = {} & |c_2|(f_2(u + T^*) - f_2(T^*))u' + |c_1|(f_1(v + C^*) - f_1(C^*))v' + Aww' \\ = {} & |c_2| f_2'(u_p)u(-m_T u + c_1(f_1(v + C^*) - f_1(C^*))) \\ & + |c_1| f_1'(v_p)v(-m_C v + c_2(f_2(u + T^*) - f_2(T^*)) + c_3(f_3(w + E^*) - f_3(E^*))) \\ & + Aw(-m_E w) \end{aligned}$$

$$= (m_T|c_2|f_2'(u_p)u^2 - |c_2|c_1f_1'(v_p)f_2'(u_p)uv - |c_1|c_2f_1'(v_p)f_2'(u_p)vu$$

$$+m_C|c_1|f_1'(v_p)v^2 - c_3|c_1|f_1'(v_p)f_3'(w_p)vw + Am_Ew^2$$

$$= (u\ \ v\ \ w)\cdot\ \boldsymbol{M}(u_p, v_p, w_p)\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where v_p, u_p and w_p are intermediate points from Mean Value Theorem and

$$M(u,v,w) = \begin{pmatrix} m_T|c_2|f_2'(u) & -\frac{1}{2}f_1'(v)f_2'(u)(|c_1|c_2 + c_1|c_2|) & 0 \\ -\frac{1}{2}f_1'(v)f_2'(u)(|c_1|c_2 + c_1|c_2|) & m_C|c_1|f_1'(v) & -\frac{1}{2}|c_1|c_3f_1'(v)f_3'(w) \\ 0 & -\frac{1}{2}|c_1|c_3f_1'(v)f_3'(w) & Am_E \end{pmatrix}$$

Condition $L'(u, v, w) > 0$ for $(u, v, w) \neq (0, 0, 0)$ is equivalent to the matrix above being positive definite. We show that $\boldsymbol{M}$ is positive definite for every (u, v, w) by applying Sylvester's criterion. The first minor is positive due to $m_T|c_2| > 0$ and (F3). The second minor

$$|c_1||c_2|m_Tm_Cf_1'(v)f_2'(u) - \frac{1}{4}f_1'(v)^2f_2'(u)^2(|c_1|c_2 + c_1|c_2|)^2$$

$$= c_1c_2f_1'(v)f_2'(u)(m_Tm_C - c_1c_2f_1'(v)f_2'(u))$$

is positive due to (F3) and the assumptions $c_1c_2 > 0$, $f_1'(v)f_2'(u) < m_Tm_C/c_1c_2$. Finally, the third minor (determinant of $M(u, v, w)$) reads

$$\begin{aligned} &Am_Em_Tm_C|c_1||c_2|f_2'(u)f_1'(v) \\ &\qquad - f_2'(u)f_1'(v)^2c_1^2|c_2|(\frac{1}{4}m_Tc_3^2f_3'(w)^2 \\ &\qquad + |c_2|f_2'(u)Am_E) \end{aligned}$$

$$\begin{aligned} &= c_1c_2f_1'(v)f_2'(u)(Am_E(m_Tm_C - c_1c_2f_1'(v)f_2'(u)) \\ &\qquad - \frac{1}{4}m_T|c_1|c_3^2f_1'(v)f_3'(w)^2) \end{aligned}$$

For chosen A, due to (F3) and the theorem assumptions, the third minor is positive. Therefore, the L function satisfies the assumptions guaranteeing the global stability of (T^*, C^*, E^*).

For Case 1C-D, we get one equilibrium point. Using the same approach as above, we can prove that it is globally stable.

Theorem 6

Let $c_1c_2 < 0$, (T^*, C^*, E^*). be a solution to System 7 i.e., it is an equilibrium point of System 5. Then System 19 has an equilibrium point (0,0,0), which is globally stable.

Proof

The proof proceeds exactly the same as the proof of Theorem 5 using Eq. 20 with

$$A = \frac{c_3^2|c_1|f_3'(0)f_1'(0)}{4m_E m_C} + 1.$$

Case 2

We know from Basic Properties that depending on the nature of the influence functions f_1 and f_2, we get different number of equilibrium points of System 5. The sign of Eq. 12 also plays an important role in Case 2. On this basis, we formulate the following lemma that allow us to evaluate the stability of any equilibrium point P_i.

Lemma 3

Let $i \in \mathbb{Z}$ be fixed. If for an equilibrium $P_i := (T_i, C_i, E_i)$ of System 5 we have

$$c_1c_2f_1'(C_i)f_2'(T_i) - m_Tm_C + m_Tc_4f_4'(C_i) < 0 \tag{21}$$

and

$$m_T + m_C - c_4f_4'(C_i) > 0, \tag{22}$$

then P_i is stable. If

$$c_1c_2f_1'(C_i)f_2'(T_i) - m_Tm_C + m_Tc_4f_4'(C_i) > 0, \tag{23}$$

then P_i is unstable.

Proof

Let Inequality 21 be satisfied and Δ be as in Eq.13. If $\Delta \geqslant 0$, then from Inequalities 21 and 22 and Vieta's formulas, we obtain that $\lambda_2, \lambda_3 < 0$. Thus, P_i is stable. For $\Delta < 0$, one gets

$$\mathrm{Re}\lambda_2 = \mathrm{Re}\lambda_3 = -\frac{1}{2}\big(m_T + m_C - c_4f_4'(C_i)\big).$$

From Inequality 22, $\mathrm{Re}\lambda_2, \mathrm{Re}\lambda_3 < 0$. Thus, P_i is stable. Note that if Inequality 23 holds, then Δ is positive. Since

$$\lambda_2, \lambda_3 = \frac{1}{2}(-(m_T + m_C - c_4f_4'(C_i)) \pm \sqrt{\Delta}),$$

$\lambda_2 > 0$ and $\lambda_3 < 0$. Consequently, P_i is unstable.

Remark 2

Note, that for $c_1c_2 > 0$, Inequality 21 implies that

$$c_4f_4'(C_i) < -\frac{c_1c_2f_1'(C_i)f_2'(T_i)}{m_T} + m_C,$$

yielding

$$m_T + m_C - c_4f_4'(C_i) > m_T + m_C + \frac{c_1c_2f_1'(C_i)f_2'(T_i)}{m_T} - m_C > 0.$$

which means that Inequality 23 is always satisfied. For $c_1c_2 < 0$, Inequality 21 is true if $c_4 f_4'(C) < m_C$, which leads to

$$m_T + m_C - c_4 f_4'(C_i) > m_T + m_C + -m_C > 0,$$

hence, Inequality 23 is always satisfied.

Theorem 7

Let $c_1c_2 < 0$, (T^*, C^*, E^*) be a solution to System 7 i.e., it is an equilibrium point of System 5. Then System 19 has an equilibrium point (0, 0, 0). Moreover this equilibrium point is globally stable if

1. $f_4'(v) < \frac{m_C}{c_4}$ $for\ all\ u \in \mathbb{R}$,
2. (T^*, C^*, E^*) $is\ a\ unique\ equilibrium\ point\ of$ $System$ 5.

Proof

We define System 19 and 20 where

$$A = \frac{|c_1|c_3^2 f_1'(0) f_3'(0)^2}{4m_E(m_C - c_4 f_{4max}')} + 1$$

and $f_{4\max}'$ is the biggest value which f_4' takes. It is obvious that $L(u, v, w) = 0$ iff $u = v = w = 0$.

We use the Mean Value Theorem to calculate the derivative of L along the trajectory of System 5:

$$L'(u,v,w) = |c_2|(f_2(u+T^*) - f_2(T^*))u' + |c_1|(f_1(v+C^*) - f_1(C^*))v' + Aww'$$

$$= |c_2|(f_2(u+T^*) - f_2(T^*))(-m_T u + c_1(f_1(v+C^*) - f_1(C^*)) + |c_1|(f_1(v+C^*) - f_1(C^*))(-m_C v + c_2(f_2(u+T^*) - f_2(T^*)) + c_3(f_3(w+E^*) - f_3(E^*)) - c_4(f_4(v+C^*) - f_4(C^*))) + Aw(-m_E w) - (m_C c_1 v^2 f_1'(v_p) - 2c_1c_2 f_2'(u_p) f_1'(v_p) - c_1c_4 f_4'(v_p) f_1'(v_p) v^2$$

$$= -(u\ \ v\ \ w) \cdot \boldsymbol{M}(u_p, v_p, w_p) \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where v_p, u_p and w_p are intermediate points from Mean Value Theorem and using the relation $|c_1|c_2 = -|c_2|c_1$. Condition $L'(u, v, w) < 0$ for $(u, v, w) \neq (0, 0, 0)$ is equivalent to the matrix above being positive definite. We define

$$M(u,v,w) = \begin{pmatrix} m_T|c_2|f_2'(u) & 0 & 0 \\ 0 & m_C|c_1|f_1'(v) - |c_1|c_4 f_4'(v) f_1'(v) & -\frac{1}{2}|c_1|c_3 f_1'(v) f_3'(w) \\ 0 & -\frac{1}{2}|c_1|c_3 f_1'(v) f_3'(w) & Am_E \end{pmatrix}$$

We show that this matrix is positive definite for every (u, v, w) by applying the Sylvester's criterion. The first minor is positive due to $m_T > 0$ and (F3). The second minor

$$m_T|c_1||c_2|f_1'(v)f_2'(u)(m_C - c_4 f_4'(c))$$

is positive due to $f_4'(v) < m_C/c_4$ and (F3). Finally, the third minor (determinant of $M(u, v, w)$) is

$$Am_E m_T|c_2|f_2'(u)|c_1|f_1'(v)(m_C - c_4 f_4'(v)) - \frac{1}{4}c_1^2 c_3^2 f_1'(v)^2 m_T|c_2|f_2'(u)c_3^2 f_3'(w)^2$$

$$= m_T c_1 c_2 f_1'(v)f_2'(u)(Am_E(m_C - c_4 f_4'(v)) - \frac{1}{4}|c_1|f_1'(v)c_3^2 f_3'(w)^2)$$

For chosen A, due to (F3) and the theorem assumptions, the third minor is positive. Therefore, if the above condition is met, the L function meets all the assumptions guaranteeing the global stability of (T^*, C^*, E^*).

Theorem 8

Let $c_1c_2 > 0$, (T^*, C^*, E^*) be a solution to System 7 i.e., it is an equilibrium point of System 5. Then System 19 has an equilibrium point (0, 0, 0). Moreover, this equilibrium point is globally stable if

1. $f_1'(v)f_2'(u) < \frac{m_T(m_C - c_4 f_4(v))}{c_1 c_2}$ $for\ all\ u \in \mathbb{R}$,
2. (T^*, C^*, E^*) $is\ a\ unique\ equilibrium\ point\ of$ $System$ 5.

Proof

The proof proceeds exactly the same as the proof of Theorem 7 using Eq. 20 with

$$A = \frac{m_T c_3^2 |c_1| f_1'(0) f_3'(0)^2}{4m_E(m_T m_C - c_1 c_2 f_1'(0) f_1'(0) f_2'(0))} + 1,$$

where $f_{4\max}'$ is the biggest value which $f_{4\max}'$ takes.

NUMERICAL SIMULATIONS

This section is dedicated to numerical simulations for System 5 with given influence functions, which will be indicated later in this paper. Moreover, it should be recalled that the basic assumption for this analysis is that the external environment is toxic and has a negative impact on the client. All simulations are performed using Maple and will be divided into cases. For Cases 1A-1D it is assumed that $f_i(x) = \arctan(x)$, for $i = 1, 2, 3$, and that each group has the same rate of mood change in the solitude ($a_T = m_C = m_E = 1$) for Cases 1A-2D.

Case 1A

In this case, we will consider a situation in which the positive influence

of a pessimistic therapist (b_T = -1) with an initial experience k_T = 1 on a neutral client (b_C = 0) is stronger than the client's influence on the therapist (c_1 =1, c_2 = 2). According to the assumption, the external environment is toxic (b_E=-1) and influences the client by parameter c_3 = 0.5. The result including the above parameters in the simulations is presented in Fig. 9.

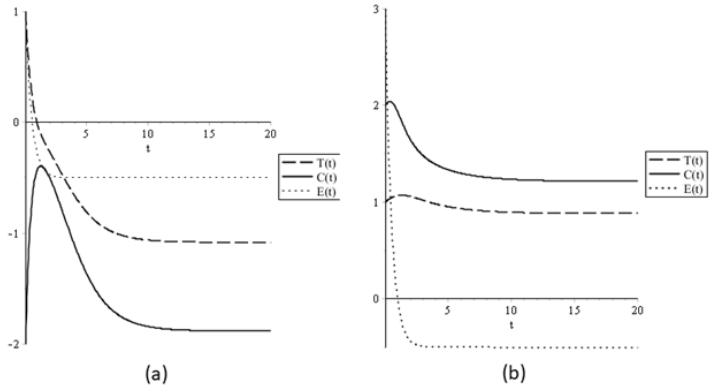


Fig. 9. Possible changes of the dynamics of the emotional states for Case 1A.

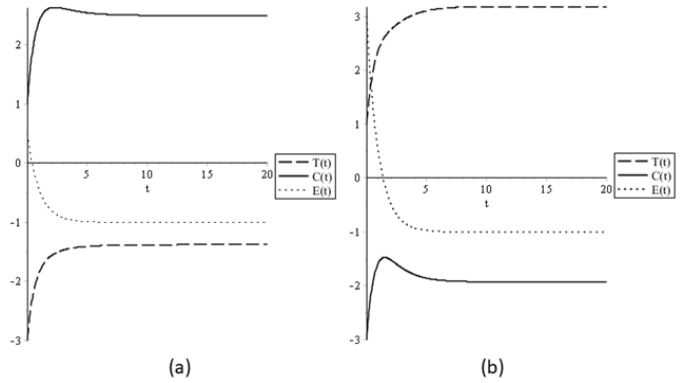


Fig. 10. Possible changes of the dynamics of the emotional states for Case 1B.

Based on the analysis presented in Existence of Equilibria, for Case 1A, we can obtain a maximum of three equilibrium points. Moreover, the simulations in Fig. 9 confirm that two of these equilibrium points are stable. By considering positive initial conditions $T(0)$ = 1, $C(0)$ = 2, $E(0)$ = 3, simulation in Fig. 9(a) shows successful end of the therapy. However, if the client begins the therapy in a negative emotional state, the therapy fails as shown in Fig. 9(b) for initial conditions $T(0)$ = 1, $C(0)$ = -2, $E(0)$ =1. It indicates that despite the initial increase in the client's emotional state, a sudden decrease is noticeable, the probable cause

of which may be a decrease in both the emotional state of the therapist and the toxic influence of the environment.

Case 1B

Now let a pessimistic therapist (b_T = -1) with an initial experience k_T = 2 and an optimistic client (b_C = 1) negatively affect each other (c_1 = 1, c_2 = -2). We assume that the parameters related to the external environment remain the same as in the simulation for Case 1A as shown in Fig. 10.

As in Case 1A, we can say that graphs in Fig. 10 present two possible - different from each other - scenarios of ending the therapy. In the first simulation, the client, unlike the therapist, reaches a positive emotional state. Despite the above, it should be treated as a therapy success, because we put the client's mental health as a priority. As shown in Fig. 10(a), such an end result of the therapy can be observed when considering the following initial conditions $T(0)$ = -3, $C(0)$ = 1, $E(0)$ = 0.5. However, when $T(0)$ = 1, $C(0)$ = -3, $E(0)$ = 3 we observe the failure of the therapy process with simultaneous increase in the emotional state of the therapist as shown in Fig. 10(b). The above can be interpreted as an increase in the therapist's experience.

Case 1C

We assume the existence of a neutral therapist (b_T = 0)with an initial experience of k_T = 1, who is exposed to a positive influence (c_1 = 2) from an optimistic client (b_C = 6). We also assume that the client is negatively influenced by both the therapist (c_2 = -6) and the external environment (b_E = -1, c_3 = 1). Simulation of solutions to System 5 for $T(0)$ = -2, $C(0)$ = 1, $E(0)$ = 0 is shown in Fig. 11(a). It indicates a turbulent relationship resulting from the therapy, ultimately leading to a result that is not obvious at first glance, consisting in a positive conclusion of the therapy process by the client. This result is related to the high level of client optimism.

Case 1D

In the next simulation analyzed, we will take into account the strongly positive impact of (c_2 = 7) of a pessimistic therapist (b_T = -3)with a fairly high experience k_T = 4 on a pessimistic client (b_C = -2) who negatively affects the therapist (c_1 = -4). Moreover, the client will be affected by the toxic environment (b_E = -4) with the power of c_3 = 4.

For the initial conditions $T(0)$ = 0, $C(0)$ = 1, $E(0)$ = 0, at the beginning of the therapy, we observe a significant deterioration of the relationship between client and therapist, which improves and stabilizes in the longer horizon as shown in Fig. 11(b). However, the high toxicity of the environment, which affects the client with significant power, makes it insufficient and the therapist ultimately fails throughout the therapy process, even though the impact was strongly positive.

Numerical simulations for Cases 2A-2D will be carried out below. From the considerations in Section 3, we know that the influence functions defined as $f_1(C) = \arctan(C)$ and $f_2(T) = \arctan(T)$ will yield the same results as in the 1A-1D

cases. Hence, for the cases discussed below, we assume that $f_3(E) = \arctan(E)$, $f_4(C) = \sin(C)$ and at least one of the influence functions f_1, f_2 will be linear. At the same time, for further simulations we assume that if one of the functions f_1, f_2 is linear, then the second function will be defined as arctan.

Case 2A

In the current case, we assume that $f_2(T) = T$. We consider a scenario in which a pessimistic therapist ($b_T = -1$) with an initial experience $k_T = 2$ has a positive influence on a pessimistic client ($b_C = -1$) with a power of $c_2 = 2$ and the emotionally unstable client ($c_4 = 5$) positively influences the therapist with a power of $c_1 = 1$. The external environment has a toxicity ($b_E = -4$) and influences the client in a negative way with a power of 3. The adoption of the above parameters creates three simulation graphs of System 5 as shown in Fig. 12.

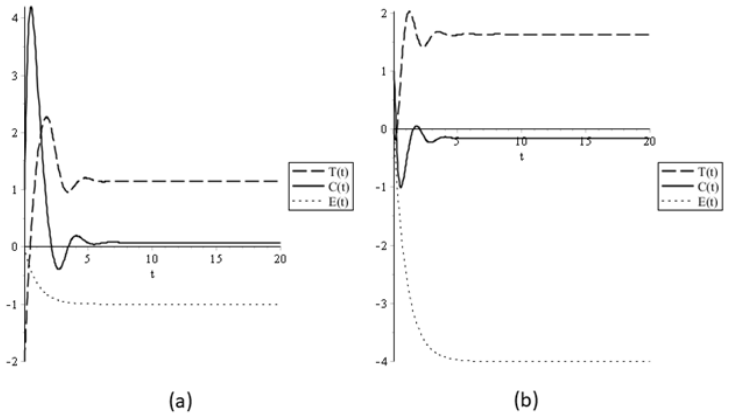


Fig. 11. Numerical simulations for Case 1C -- (a) and Cases 1D -- (b).

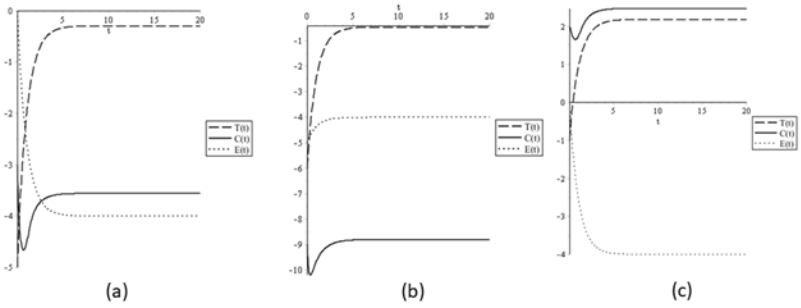


Fig. 12. Possible changes of the dynamics of the emotional states for Case 2A.

Based on the considerations in Section 3 for cases where only one of the functions f_1, f_2 is given as a linear function, we can obtain finite number of

equilibrium points. Moreover, the simulations indicated in Fig. 12 show that three of these equilibrium points are stable. These simulations present possible end of therapy scenarios, but only one of them seems to be in line with our intuition. The first two graphs Fig. 12(a) and (b) are obtained for the initial conditions $T(0) = -5$, $C(0) = -3$, $E(0) = 0$ and $T(0) = -6$, $C(0) = -9$, $E(0) = -5$. In these cases, the therapy ends negatively, although the relationship between the client and the therapist was positive. However, it should be pointed out that it is possible to successfully complete the therapy as long as the initial positive attitude of both parties to the therapy is appropriately positive as shown in Fig. 12(c). In the analyzed case $T(0) = -1$, $C(0) = 2$, $E(0) = 0$. The need to adopt the right attitude may be caused by the strong influence of the external environment and not enough experience of the therapist.

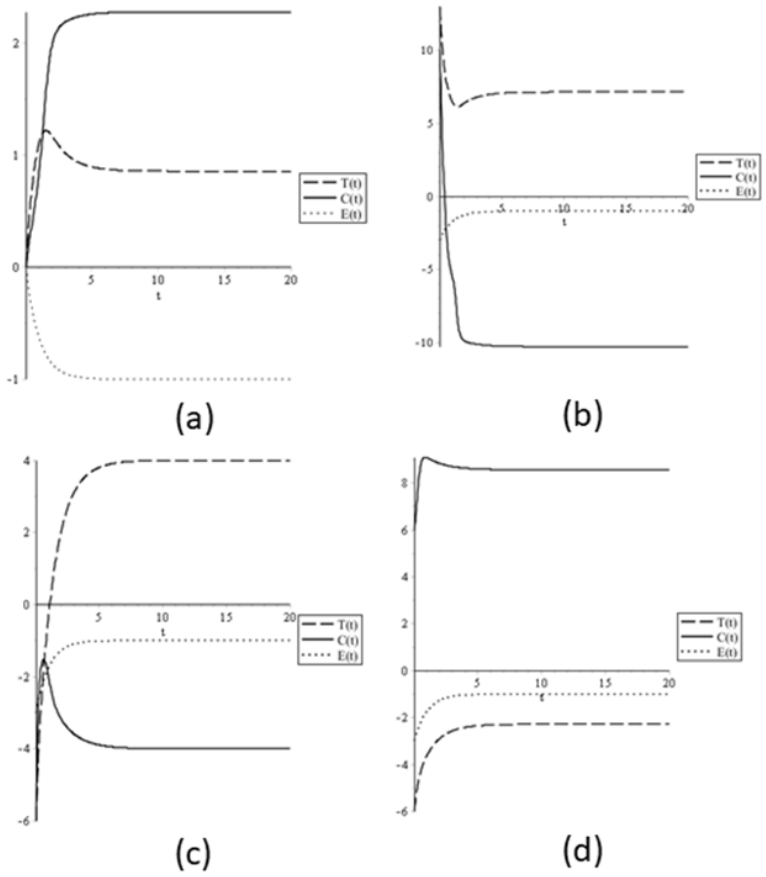


Fig. 13. Possible changes of the dynamics of the emotional states for Case 2B.

Case 2B

Let f_1 and f_2 be linear functions. Now, we consider a situation in which an optimistic ($b_C = 1$), an unstable client ($c_4 = 5$) has a negative influence ($c_2 = -2$) on a pessimistic therapist ($b_T = -1$) with average experience $k_T = 3$. We also assume that the external environment is toxic ($b_E = -1$) and that it acts with a power equal to 1. The client influences the therapist with the power of $c_1 = -0,5$. For such adopted parameters, we obtain the solutions presented in Fig. 13.

In this case, it is possible to obtain an infinite number of solutions, and the presented simulation graphs indicate a convergence of up to four selected emotional states. The initial conditions for the successive simulations as shown in Fig. 13(a) - (d) are: $T(0) = 0$, $C(0) = 0$, $E(0) = 0$; $T(0) = 13$, $C(0) = 10$, $E(0) = -3$; $T(0) = -6$, $C(0) = -6$, $E(0) = -3$ and $T(0) = -6$, $C(0) = 6$, $E(0) = -3$. Note that none of the simulations presented in Fig. 13 shows the simultaneous negative culmination of therapy for the therapist and the client. Moreover, one of them represents a positive end to the therapy for both sides. It is possible to conclude that the number of different results of the therapy process is caused by the client's high emotional instability and is strongly dependent on the initial attitude of the parties to the therapy.

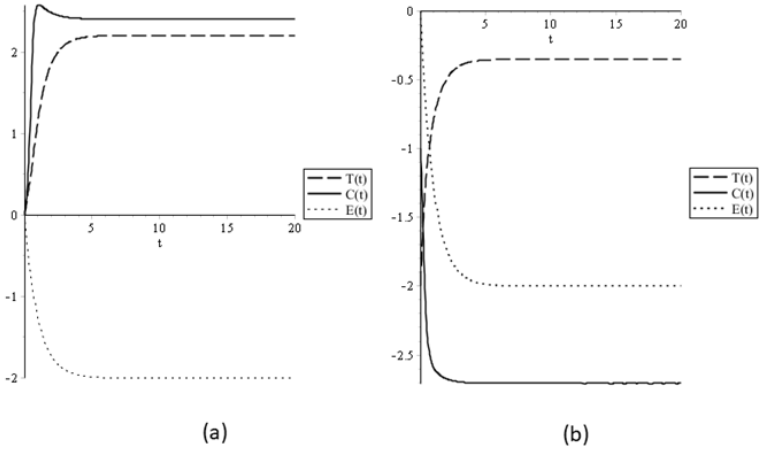


Fig. 14. Possible changes of the dynamics of the emotional states for Case 2C.

Case 2C

Let f_1 and f_2 be linear functions. Numerical simulations will be carried out showing the final effects of therapy, in which an optimistic client ($b_C = 1$), with emotional instability $c_4 = 7$ has a negative influence on the therapist ($c_2 = -1$), who is neutral ($b_T = 0$) and has initial experience $k_T = 1$. In addition, the client has a positive influence on the therapist ($c_1 = -0,5$), while the external environment is toxic ($b_E = -2$) and affects the client negatively with the power of $c_3 = 1$. The above parameters form the following simulation graphs of System 5 as shown in Fig. 14.

Figures 14(a) and (b) show the varying final effects of therapy for initial conditions respectively $T(0) = 0$, $C(0) = 0$, $E(0) = 0$ and $T(0) = -2$, $C(0) = -1$, $E(0) = 0$. One might expect that the client should end the therapy negatively, as opposed to the therapist. However, they both end the therapy with either a positive or a negative emotional state. Such a result may depend on the initial approach of the parties towards each other, the power of mutual interactions between the client and the therapist and the influence of the toxic external environment.

Case 2D

Let $f_2(T) = T$. In the last considered situation, an optimistic ($b_T = 2$) therapist with no experience ($k_T = 0$) has a positive influence ($c_2 = 1$) on an optimistic ($b_C = 1$), emotionally unstable ($c_4 = 5$) client. The client has a negative influence on the therapist with the power of $c_1 = -2$. Besides, the external environment is toxic ($b_E = -4$) and influences the client with a power equal to 2 as shown in Fig. 15. Simulation of solution to System 5 for $T(0) = 1$, $C(0) = -2$, $E(0) = 3$ is shown in Fig. 15(a). It indicates that the therapist ends the therapy with a positive emotional state despite the negative influence of the client, whereas the client ends the therapy negatively. By considering initial conditions $T(0) = 1$, $C(0) = 0$, $E(0) = 1$, in Fig. 15(b) it is shown that the therapist ended the therapy in a negative way with a positive conclusion of the therapy for the client. The above does not seem to be necessarily in line with the initial intuition for completing this therapy.

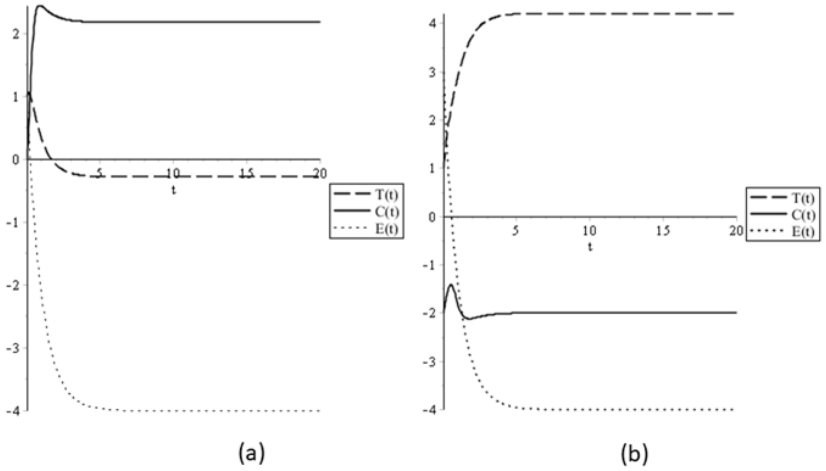


Fig. 15. Possible changes of the dynamics of the emotional states for Case 2D.

CONCLUSION

Overall, the changes we proposed in this article into the already known model of dyadic interactions allowed us to adjust this model to the situation of a person with mental problems seeking help in psychotherapy. It led to the

formulation of a new equation describing the external environment, the addition of a function characterizing the client's emotional instability, which can be related to mental comorbidities and the inclusion of a coefficient representing the therapist's experience. The therapy process description is based on observations from many publications in the field of psychology and psychotherapy. Thanks to simulations conducted to examine the unfavorable impact of the external environment, various therapy scenarios emerged, depending on the mutual interactions of the client and the therapist and the emotional stability/instability of the client. One of the most important findings is that the client's emotional instability plays a key role in shaping potential therapy outcomes. The more emotionally unstable the client, the greater the variety of possible therapy outcomes. This highlights the importance of tailoring therapeutic approaches to each person's unique emotional needs and sensitivities. It should be noted that the work on this model is not yet complete and there is a scope for further research and improvement. One potential area for future considerations is the inclusion of a delay factor reflecting the client's response time to therapist interventions during the therapy process. Such time dimension could provide valuable information about the dynamics of the therapeutic relationship and its impact on the overall effectiveness of psychotherapy.

In conclusion, the developed model of psychotherapy, based on the Liebovitch's et al. (2011), has shown promise in explaining the intricate dynamics of the therapeutic process. By taking into account factors such as external influences, the client's emotional instability, and the therapist's experience, this model allows for a more complete understanding of the multifaceted nature of psychotherapy. Continued research in this area may help therapists customize therapy to better meet clients' needs.

DISCUSSION

One of the crucial aspects of modeling in any real phenomena, including in modeling psychotherapy, is testing and validation. While studying emotions may appear difficult, many researchers endeavor to explore them. An early experiment conducted by Gottman et al. (1990) and (2002), a renowned psychologist specializing in relationships, involved exploring the emotional dynamics of 100 married couples. Gottman and co-authors established a specialized observation setup known as the "Love Lab" to capture emotional responses. Couples participated in various activities within this environment, including discussions, problem-solving tasks, and shared leisure time, whereas everything was recorded for later analysis. During these interactions, researchers examined diverse behavioral facets such as gestures, vocal intonations and facial expressions to gauge the emotional states of individuals. They analyzed this data to discern patterns capable of predicting potential marital challenges, such as communication issues or even divorce. Subsequently, they formulated several pivotal indicators that could influence marital stability or instability. On base insights gleaned in psychoanalysis of human behavior, contemporary endeavors necessitate innovative approaches to analyzing emotional states within our model.

In Mieskes and Stiegelmayr (2018), authors adopt a similar approach by recording therapy sessions and deriving insights from the recordings. This method allows for a comprehensive examination of therapeutic interactions and facilitates the extraction of valuable data for analysis and interpretation.

While recording therapy sessions poses logistical challenges, alternative methodologies are feasible. One such approach involves surveying therapy participants, utilizing a numerical scale to assign emotional levels. For instance, −15 might denote profound sadness and anger, 0 represents neutrality, and 15 signifies happiness. These surveys, along with ensuing results, would be curated and assessed by psychotherapy specialists.

Numerous studies have embraced the approach of utilizing modern technologies for data collection, as highlighted in various works. Laver, Mcaleavey, Valaker, Castonguay, and Moltu (2024) emphasized the need to collect data from therapeutic sessions based on observations from multiple sources. For example, in the study reported in Boswell, Kraus, Miller, and Lamber (2015), researchers detailed pre- and post-session data collection using read-only memory (ROM) to store information. Nevertheless, the authors admit that for some clients, the use of new technologies to collect data may prove too time-consuming, which suggests continuing to use traditional "paper and pencil" methods. Moreover, in Reynolds, Jr., and Stilles (2007) the use of online forms as a real alternative to paper surveys was analyzed, aligning with the modernization of data collection practices. However, several studies addressing this subject matter highlight a recurring challenge: clients reluctance to provide feedback during the therapeutic process.

Nowadays, the use of modern technologies such as smartphones offers a new approach to collecting customer data, especially through the use of sensors, as described in Harari et al. (2016). In this context, smartphones can serve as powerful tools for collecting behavioral data about their owners. For instance, sensors can capture information regarding sleep patterns, phone usage duration, travel history, and even social interactions through metrics like call frequency, text message volume, or interactions on social media applications. This method of data collection holds the promise of providing more accurate determinations of coefficients such as m_C, m_T, b_C, and b_T.

Alternatively, leveraging advancements in technology, such as headbands capable of detecting brainwave patterns associated with eight distinct emotions, offers another opportunity. Specialists can analyze these outcomes to ascertain emotional states of clients, therapists, or individuals within their immediate environment. This underscores the feasibility of testing and validating our model, however requiring concerted effort from diverse scientific disciplines.

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